Near-best bivariate spline quasi-interpolants on a four-directional mesh of the plane ⋆

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Abstract

Spline quasi-interpolants (QIs) are practical and effective approximation operators. In this paper, we construct QIs with optimal approximation orders and small infinity norms called near-best discrete and integral quasi-interpolants which are based on Ω-splines, i.e. B-splines with regular lozenge supports on the uniform four directional mesh of the plane. These quasi-interpolants are obtained so as to be exact on some space of polynomials and to minimize an upper bound of their infinity norms which depend on a finite number of free parameters. We show that this problem has always a solution, which is not unique in general. Concrete examples of these types of quasi-interpolants are given in the last section.

Key words: Ω-splines, discrete quasi-interpolants, integral quasi-interpolants, near-best quasi-interpolants
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1 Introduction

Let τ be the uniform triangulation of $\mathbb{R}^2$ whose set of vertices is $\mathbb{Z}^2 \cup \left(\mathbb{Z}+\frac{1}{2}\right)^2$, and whose edges are parallel to the four directions $e_1 = (1,0)$, $e_2 = (0,1)$,

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$e_3 = (1, 1)$ and $e_4 = (-1, 1)$. Let $\mathbb{P}_n$ be the space of bivariate polynomials of total degree at most n, and let $\mathbb{P}_n^k(\tau)$ be the space of piecewise polynomial functions of total degree n and class C^k defined on τ.

In this paper, we are interested in the construction of quasi-interpolants based on Ω-splines, i.e. B-splines with regular lozenge supports. The family of Ω-splines contains the classical box-splines in $\mathbb{P}_{4k}^{3k-1}(\tau)$, $k \geq 1$ (see [4], [6]), together with new families of B-splines defined and studied in [9], [10] and [13].
For a given Ω-spline ϕ, we denote $\mathcal{S}(\phi) = \{\sum_{\alpha \in \mathbb{Z}^2} c(\alpha)\, \phi(\cdot - \alpha),\ c(\alpha) \in \mathbb{R}\}$ the space of splines generated by the family of integer translates $\mathcal{B}(\phi) = \{\phi(\cdot - \alpha),\ \alpha \in \mathbb{Z}^2\}$, and by $\mathbb{P}(\phi)$ the space of polynomials of maximal total degree included in $\mathcal{S}(\phi)$. We study new families of discrete or integral quasi-interpolants in the space $\mathcal{S}(\phi)$ which are exact on $\mathbb{P}(\phi)$, and minimize a simple upper bound of their uniform norms. These quasi-interpolants are the natural extensions to the bivariate case of those introduced in [2] and are similar to those given in [3] (see also [7]). They are written in the form $Qf = \sum_{\alpha \in \mathbb{Z}^2} \lambda_\alpha(f)\, \phi(\cdot - \alpha)$, where $\lambda_\alpha(f)$ is a linear combination of values $f(\beta)$ or of mean values $< f, \phi(\cdot - \beta) >= \int_{\mathbb{R}^2} f(x)\, \phi(x - \beta)\, dx$, with $\beta \in \mathbb{Z}^2$ lying in some lozenge centered at $\alpha \in \mathbb{Z}^2$. Many authors have already considered these types of operators (see [6], [4]), but the idea to minimize the norm is original and gives raise to new families of quasi-interpolants which seem interesting.

The paper is organized as follows. In Section 2, we recall some results on Ω-splines and lozenge sequences. Then, in Section 3, we introduce discrete and integral quasi-interpolants (QIs) based on some Ω-spline ϕ and exact on $\mathbb{P}(\phi)$. Starting from these QIs, we study in Section 4 new families of QIs. They are obtained by solving a minimization problem that admits always a solution which is not unique in general. Finally, in Section 5, we give two examples of each type of these operators.

2 Ω-splines, symmetrical lozenge sequences and difference operators

2.1 Ω-splines

For $k, l \in \mathbb{N}$ we denote by $\Omega_{k,l}$ the octagon whose sides parallel to e_1 and e_2 (resp. e_3 and e_4) are composed of k (resp. l) edges of τ. For $k = l = 0$, we put $\Omega_{0,0} = \{0\}$.
Starting from simple Ω-splines of class C^r, $r \geq -1$, supported respectively on $\Omega_{1,0}$ and $\Omega_{0,1}$, the authors in [9], and [12] have constructed composed Ω-splines with supports $\Omega_{k,l}$, $k, l \geq 1$. The classical box-splines in τ are important

examples of such splines. The simple Ω-splines are noted σ_r and λ_r, and are characterized by the following theorem (see [10] and [11]).

Theorem 1 *Up to a multiplicative constant, there exist unique Ω-splines σ_r and λ_r of class C^r, of minimal degree $d_r = 2r+1$ for r even and $d_r = 2r+2$ for r odd. Moreover, the integer translates of the two functions do not form a partition of unity.*

Now, by convolution of these simple B-splines with convolution powers of the characteristic functions σ of $\Omega_{1,0}$ and λ of $\Omega_{0,1}$, we can construct different families of Ω-splines on τ. Indeed, let σ^k (resp. λ^l) be the $k-th$ (resp. $l-th$) convolution power of σ (resp. λ), and define the box-splines $\omega^{k,0} = \sigma^k$, $\omega^{0,l} = \lambda^l$ and $\omega^{k,l} = \sigma^k * \lambda^l$ for $k, l \geq 1$. Then the composed Ω-splines that we consider here are defined as follows: $\sigma_r^{0,0} = \sigma_r$, $\lambda_r^{0,0} = \lambda_r$ and for $(k,l) \neq (0,0)$, $\sigma_r^{k,l} = \sigma_r * \omega^{k,l}$, $\lambda_r^{k,l} = \lambda_r * \omega^{k,l}$. For $r = -1$, these Ω-splines are classical box-splines. Their principal properties are summarized in the following theorem, for more details see [9] and [12].

Theorem 2 *(i) The support of $\sigma_r^{k,l}$ (resp. $\lambda_r^{k,l}$) is the octagon $\Omega_{k+1,l}$ (resp. $\Omega_{k,l+1}$).*

(ii) $\sigma_r^{k,l}$ and $\lambda_r^{k,l}$ are positive B-splines of degree $2(k+l+r)+1$ for r even and $2(k+l+r)+2$ for r odd. They are of class $2k+l+r$ for $l \geq k$ and $k+2l+r$ for $k \geq l$.

(iii) For $r = -1$, $\mathbb{P}\left(\sigma * \omega^{k,l}\right) = \mathbb{P}\left(\lambda * \omega^{k,l}\right) = \mathbb{P}_{2k+l}$. For $r \geq 0$, $\mathbb{P}\left(\sigma_r^{k,l}\right) = \mathbb{P}\left(\lambda_r^{k,l}\right) = \mathbb{P}\left(\omega^{k,l}\right) = \mathbb{P}_{2k+l-1}$.

(iv) For $\phi = \sigma_r^{k,0}$ or $\phi = \lambda_r^{k,0}$, $r, k \geq 0$, the family $\mathcal{B}(\phi)$ is globally linearly independent, i.e. $\sum_{\alpha \in \mathbb{Z}^2} c(\alpha)\,\phi(\cdot - \alpha) = 0$ implies $c(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^2$.

(v) For $\phi = \sigma_r^{k,l}$ or $\phi = \lambda_r^{k,l}$, $r, k \geq 0$ and $l \geq 1$, $\mathcal{B}(\phi)$ is globally linearly dependent. The dependence relation is given by $\sum_{\alpha \in \mathbb{Z}^2} (-1)^\alpha \phi(\cdot - \alpha) = 0$.

In what follows, we denote by ϕ one of the previous Ω-splines. Property (iii) gives the approximation order of a smooth function in the space $\mathcal{S}(\phi)$. Then it is natural to construct spline operators which achieve this order of approximation. In the literature, there exist many methods to do this. For example, in [4] and [6] are described quasi-interpolants using Appell sequences, Neumann series or Fourier transform. In [9] and [13] (see also [14]), discrete and integral quasi-interpolants are defined from the values of an Ω-spline on a four direction mesh by exploiting the relation between lozenge sequences and central difference operators. It seems that this latter method is best adapted for the study proposed here. So, we recall in the following subsections some properties of lozenge sequences and of the associated algebra of difference operators introduced in [13].

Let Ω be the support of a given Ω-spline ϕ, and Λ_p, $p \in \mathbb{N}$, a largest lozenge contained in Ω. We denote by $\mathcal{L}_p$ the vector space of real sequences $a = \left\{a(\alpha), \alpha \in \mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2\right\}$ having their support in Λ_p, i.e., satisfying $a(\alpha) = 0$ for all $\alpha \notin \Lambda_p^* = \Lambda_p \cap \left(\mathbb{Z}^2 \cup \left(\mathbb{Z} + \frac{1}{2}\right)^2\right)$, and which are invariant by the group of symmetries and rotations of the lozenge Λ_p. It is easy to prove the following result.

Theorem 3 *It holds*

$$\dim \mathcal{L}_p = \begin{cases} (q+1)^2 & \text{if } p = 2q, \\ (q+1)(q+2) & \text{if } p = 2q+1. \end{cases}$$

Consequently, with any sequence $a \in \mathcal{L}_p$, we associate a list $\tilde{a} = [a_{\alpha_1}, \cdots, a_{\alpha_n}]$, where $n = \dim \mathcal{L}_p$. The correspondence between the list $\tilde{a}$ and the sequence a is described in Figure 1 for $p = 3$, so $n = 4$.

$$\begin{array}{ccccccc} & & & a_{30} & & & \\ & & a_{21} & a_{20} & a_{21} & & \\ & a_{21} & a_{11} & a_{10} & \underline{a_{11}} & \underline{a_{21}} & \\ a_{30} & a_{20} & a_{10} & \underline{a_{00}} & \underline{a_{10}} & \underline{a_{20}} & \underline{a_{30}} \\ & a_{21} & a_{11} & a_{10} & a_{11} & a_{21} & \\ & & a_{21} & a_{20} & a_{21} & & \\ & & & a_{30} & & & \end{array} \qquad \tilde{a} = [a_{00}, a_{10}, a_{20}, a_{30}, a_{11}, a_{21}]$$

Fig. 1. The relationship between the list $\tilde{a}$ and the sequence a for $p = 3$.

Let $\delta_1 \in \mathcal{L}_1$ and $\delta_2 \in \mathcal{L}_2$ be two lozenge sequences associated respectively with the lists $\tilde{\delta_1} = [-4, 1]$ and $\tilde{\delta_2} = [-4, 0, 0, 1]$. We denote by $I \in \mathcal{L}_0$, the sequence associated with the list reduced to $[1]$. For $p \geq 0$, let $T_p = \{(m,n) \in \mathbb{N}^2,\ 0 \leq m + 2n \leq p\}$ and $\mathcal{B}_p = \{\delta_1^m \delta_2^n,\ (m,n) \in T_p\}$, where the products are convolution products, i.e., the elements δ_1^m, δ_1^n and $\delta_1^m \delta_2^n$ of the spaces $\mathcal{L}_m$, $\mathcal{L}_n$ and $\mathcal{L}_{m+2n}$ respectively are given by

$$\delta_1^m = \{\delta_1^m(j) : \delta_1^1(j) = \delta_1(j),\ j \in L_1^*;\ \delta_1^m(j) = \sum_{i \in L_1^*} \delta_1(i)\, \delta_1^{m-1}(j-i),\ j \in L_m^*\},$$

$$\delta_2^n = \{\delta_2^n(j) : \delta_2^1(j) = \delta_2(j),\ j \in L_2^*;\ \delta_2^n(j) = \sum_{i \in L_2^*} \delta_2(i)\, \delta_2^{n-1}(j-i),\ j \in L_{2n}^*\},$$

and

$$\delta_1^m \delta_2^n = \{\delta^{m,n}(j) : \delta^{m,n}(j) = \sum_{i \in L_m^*} \delta_1^m(i)\, \delta_2^n(j-i)\}.$$

Then, it is easy to check that $\dim \mathcal{L}_p = \operatorname{card} \mathcal{B}_p$ and, by induction on p, one can prove that $\mathcal{B}_p$ is a basis for the space $\mathcal{L}_p$.

Remark 4 *There exist other bases for the space $\mathcal{L}_p$ similar to $\mathcal{B}_p$. Indeed, if we denote by δ_3, δ_4, δ_5 the lozenge sequences in $\mathcal{L}_2$ associated respectively with the lists $\tilde{\delta_3} = [-4, 0, 1, 0]$, $\tilde{\delta_4} = [-60, 16, 0, -1]$, $\tilde{\delta_5} = [20, -4, -1, 0]$, then it is easy to verify that each one of the families $\mathcal{B}_{i,p}$, $3 \leq i \leq 5$, defined by $\mathcal{B}_{i,p} = \{\delta_1^m \delta_i^n, (m,n) \in T_p\}$ forms a basis for the space $\mathcal{L}_p$.*

2.3 The algebra of difference operators

To the above lozenge sequences δ_1 and δ_2 we associate the difference operators Δ_1 and Δ_2 defined, for $k = 1$ or 2, by

$$(\Delta_k f)(x) = f(x + ke_1) + f(x + ke_2) - 4f(x) + f(x - ke_1) + f(x - ke_2),$$

which stands for the discrete schemes of the Laplacian operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. More precisely, the relation between these two lozenge sequences and the difference operators Δ_1 and Δ_2 is given by the following identity

$$(\Delta_k f)(\alpha) = (\delta_k * \mathsf{f})(\alpha),$$

where f denotes here the sequence $\{f(\alpha), \alpha \in \mathbb{Z}^2\}$. Furthermore, if we denote by $\mathcal{K}_p$, $p \geq 0$, the space with basis $\{\Delta_1^m \Delta_2^n, (m,n) \in T_p\}$, then it is clear that the two spaces $\mathcal{K}_p$ and $\mathcal{L}_p$ are isomorphic. On the other hand, it is simple to see that each element D of $\mathcal{K}_p$, $p \geq 0$, has a lozenge support. Then, its inverse D^{-1} in the convolution algebra $l^1(\mathbb{Z}^2)$ has a non bounded support. However, we show in the following lemma that D^{-1} is finite when restricted to some space of polynomials.

Lemma 5 *Let $k \in \mathbb{N}^*$ and $D = \sum_{(m,n) \in T_p} \alpha(m,n)\, \Delta_1^m \Delta_2^n \in \mathcal{K}_p$. Then the inverse D^{-1} of D restricted to the space $\mathbb{P}_{2k+1}$ is given by*

$$D^{-1} = \sum_{r+s \leq k} \beta(r,s)\, \Delta_1^r \Delta_2^s,$$

where $\beta(r,s)$ are solutions of the following linear system

$$\sum_{r+m \leq u, s+n \leq v} \alpha(m,n)\, \beta(r,s) = \begin{cases} 1 \text{ for } (u,v) = (0,0), \\ 0 \text{ for } (u,v) \neq (0,0). \end{cases}$$

PROOF. It derives from the fact that $\Delta_1^m \Delta_2^n p = 0$ for all $p \in \mathbb{P}_{2r-1}$ such that $m + n = r \geq 1$, and the degree $2r - 1$ is maximal. $\square$

3 Quasi-interpolants based on Ω-splines

Let ϕ be a Ω-spline. In this section we introduce new families of discrete and integral quasi-interpolants based on ϕ. They are constructed by solving minimization problems under some linear constraints. In order to give the explicit formulae of these linear constraints, it is necessary to express all the monomials of $\mathbb{P}(\phi)$ as linear combinations of integer translates of ϕ. To do this, we need some results concerning differential quasi-interpolants (see [5]).

3.1 Differential quasi-interpolants (DQIs)

For a given Ω-spline ϕ of support Ω, we denote by $\hat{\phi}$ its Fourier transform. As $\hat{\phi}(0) = 1$, we have in some neighbourhood of the origin

$$\frac{1}{\hat{\phi}(y)} = \sum_{\alpha \in \mathbb{N}^2} a_\alpha y^\alpha.$$

Let d be the integer such that $\mathbb{P}_d = \mathbb{P}(\phi)$ and $m_\alpha(x) = x^\alpha$ are the monomials of $\mathbb{P}(\phi)$. The differential operator associated with ϕ is defined by

$$\mathbb{D}f = \sum_{|\alpha| \leq d} (-\mathbf{i})^{|\alpha|} a_\alpha D^\alpha f, \quad \text{with } \mathbf{i} \text{ the complex such that } \mathbf{i}^2 = -1.$$

Let $Sf = \sum_{i \in \mathbb{Z}^2} f(i)\, \phi(\cdot - i)$ be the classical Schoenberg operator. It is well known (see e.g. [9] and [13]) that S is an automorphism on $\mathbb{P}(\phi)$ and satisfies

$$Sm_\alpha = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!} (-\mathbf{i}D)^\beta\, \hat{\phi}(0)\, D^\beta m_\alpha, \quad \text{and} \quad S^{-1} m_\alpha = g_\alpha \quad \text{for all} \quad \alpha \in \Gamma_\phi,$$

where $\Gamma_\phi = \{\alpha \in \mathbb{N}^2,\ m_\alpha \in \mathbb{P}(\phi)\}$ and g_α is the recursive family of polynomials defined by

$$g_0 = m_0, \qquad g_\alpha = m_\alpha - \sum_{j \in \mathbb{Z}^2} \phi(j) \sum_{\beta \leq \alpha, \beta \neq \alpha} \frac{(-j)^{\alpha - \beta}\, \alpha!}{(\alpha - \beta)!} g_\beta. \tag{1}$$

Moreover, we have the following result.

Lemma 6 *The operator $\mathbb{D}$ coincides on $\mathbb{P}(\phi)$ with S^{-1}. Therefore $\mathbb{D}$ is also an automorphism on $\mathbb{P}(\phi)$.*

PROOF. Consider the power series expansion $\hat{\phi}(y) = \sum_{\beta\in\mathbb{N}^2} \frac{1}{\beta!} D^\beta \hat{\phi}(0) y^\beta$. Thus, $\hat{\phi}\hat{\phi}^{-1} = 1$ implies that

$$\sum_{\alpha+\beta=\gamma} \frac{c_\alpha}{\beta!} D^\beta \hat{\phi}\,(0) = \delta_{0\gamma} = \begin{cases} 1 \text{ when } \gamma = 0, \\ 0 \text{ when} \gamma \neq 0. \end{cases}$$

On the other hand, for all $\alpha \in \Gamma_\phi$ we have

$$\begin{aligned} m_\alpha &= \sum_{\gamma\leq\alpha} (-\mathbf{i}D)^\gamma\, m_\alpha \delta_{0\gamma} = \sum_{\gamma\leq\alpha} (-\mathbf{i}D)^\gamma\, m_\alpha \sum_{\beta+\theta=\gamma} \frac{a_\beta}{\theta!} D^\theta \hat{\phi}\,(0) \\ &= \sum_{\beta,\theta\leq\alpha} (-\mathbf{i}D)^{\beta+\theta}\, m_\alpha \frac{a_\beta}{\theta!} D^\theta \hat{\phi}\,(0) = \sum_{\theta\leq\alpha} \left(\sum_{\beta\in\Gamma(\phi)} a_\beta \,(-\mathbf{i}D)^\beta \,(D^\theta m_\alpha) \right) \frac{(-\mathbf{i}D)^\theta\, \hat{\phi}\,(0)}{\theta!} \\ &= \sum_{\theta\leq\alpha} \mathbb{D}\left(D^\theta m_\alpha\right) \frac{(-\mathbf{i}D)^\theta\, \hat{\phi}\,(0)}{\theta!} = \mathbb{D}\left(\sum_{\theta\leq\alpha} D^\theta m_\alpha \frac{(-\mathbf{i}D)^\theta \hat{\phi}(0)}{\theta!} \right) \\ &= \mathbb{D} S m_\alpha. \end{aligned}$$

Then, we deduce that $\mathbb{D} = S^{-1}$ on $\mathbb{P}\,(\phi)$ and consequently $\mathbb{D}$ is an automorphism on that space. $\square$

Now, using the operator $\mathbb{D}$, we define the following differential quasi-interpolant

$$\mathcal{D}f = S\mathbb{D}f = \sum_{j\in\mathbb{Z}^2} \left(\sum_{|\alpha|\leq d} (-\mathbf{i})^{|\alpha|}\, a_\alpha D^\alpha f\,(j) \right) \phi\,(\cdot - j)\,.$$

Thus, it is clear that $\mathcal{D}$ is exact on $\mathbb{P}_d$.
According to Section 2, the space $\mathbb{P}_d$ coincides with $\mathbb{P}_{2k+l}$ when ϕ is a box-spline in $\mathbb{P}^{s(k,l)}_{2(k+l)}(\tau)$, where

$$s(k,l) = \begin{cases} 2k + l - 1 \text{ if } l > k, \\ k + 2l - 1 \text{ if } k > l. \end{cases}$$

In this case, the Fourier transform $\hat{\phi}$ is well known and the computation of the coefficients c_α can be done directly. Therefore, as

$$\mathcal{D}m_\alpha = m_\alpha \text{ for all } \alpha \in \mathbb{P}_{2k+1},$$

we easily deduce the needed expressions of m_α.
For a Ω-spline ϕ which is not a box-spline, we have not in general the explicit formula of its Fourier transform. However, as shown in the following result, the associated coefficients a_α are determined only in terms of the values $\phi\,(j)$, $j \in \operatorname{supp}(\phi) \cap \mathbb{Z}^2$, which can be computed by standard convolution algorithms (see e.g. [9]).

Lemma 7 *For any $\alpha \in \Gamma_\phi$, we have*

$$a_\alpha = \mathbf{i}^{|\alpha|} g_\alpha (0) .$$

PROOF. It derives from the fact that $g_\alpha = S^{-1} m_\alpha = \mathbb{D} m_\alpha$, for all $\alpha \in \Gamma_\phi$. $\square$

3.2 Discrete quasi-interpolants (dQIs)

Let $\Phi = \{\phi(\alpha), \alpha \in \Lambda_k^* = \Lambda_k \cap \mathbb{Z}^2\}$ be the lozenge sequence of $\mathcal{L}_k$ associated with the Ω-spline ϕ, and let $D \in \mathcal{K}_k$ be its corresponding difference operator. As the above Schoenberg operator S is an automorphism on $\mathbb{P}(\phi)$, there exists for each $p \in \mathbb{P}(\phi)$ a unique $q \in \mathbb{P}(\phi)$ such that $p = Sq$. Then, according to the definition of S, we obtain

$$\begin{aligned} Sp &= \sum_{i \in \mathbb{Z}^2} Sq(i) \phi(\cdot - i) = \sum_{i \in \mathbb{Z}^2} \left(\sum_{\alpha \in \mathbb{Z}^2} q(\alpha) \phi(i - \alpha) \right) \phi(\cdot - i) \\ &= \sum_{i \in \mathbb{Z}^2} \left(\sum_{\alpha \in \Lambda_k^*} \phi(\alpha) q(i + \alpha) \right) \phi(\cdot - i) = \sum_{i \in \mathbb{Z}^2} Dq(i) \phi(\cdot - i) . \end{aligned}$$

On the other hand, using the fact that

$$\sum_{i \in \mathbb{Z}^2} \Delta_r q(i) \phi(\cdot - i) = \sum_{i \in \mathbb{Z}^2} q(i) \Delta_r \phi(\cdot - i), r = 1, 2,$$

we deduce that

$$Sq = \sum_{i \in \mathbb{Z}^2} Dq(i) \phi(\cdot - i) = \sum_{i \in \mathbb{Z}^2} q(i) D\phi(\cdot - i) = DSq = Dp.$$

Hence, S coincides with D on $\mathbb{P}(\phi)$.
Now, if we set D^{-1} the inverse of D on $\mathbb{P}(\phi)$, then the discrete quasi-interpolant defined by

$$Qf = SD^{-1} f = \sum_{i \in \mathbb{Z}^2} D^{-1} f(i) \phi(\cdot - i) = \sum_{i \in \mathbb{Z}^2} f(i) \left(D^{-1} \phi \right) (\cdot - i) = D^{-1} Sf$$

is exact on $\mathbb{P}(\phi)$.
According to Lemma 5, the restriction to $\mathbb{P}(\phi)$ of D^{-1} is finite, and it can be written in the form

$$D^{-1} f = \sum_{\alpha \in \Lambda_k^*} a_\alpha f(\cdot + \alpha) .$$

Therefore, the above expression of Qf becomes

$$Qf = \sum_{i \in \mathbb{Z}^2} \left(\sum_{\alpha \in \Lambda_k^*} a_\alpha f(i+\alpha) \right) \phi\left(\cdot - i\right),$$

which is equivalent to

$$Qf = \sum_{i \in \mathbb{Z}^2} f\left(i\right) L\left(\cdot - i\right),$$

where L denotes the fundamental function defined by

$$L = \sum_{\alpha \in \Lambda_k^*} a_\alpha \phi((\cdot - \alpha)).$$

It is easy to verify that

$$\left\| Q \right\|_\infty \leq \nu\left(a\right) = \sum_{\alpha \in \Lambda_k^*} \left| a_\alpha \right|.$$

3.3 Integral quasi-interpolants (iQIs)

It was shown in [9] and [12], that each Ω-spline ϕ considered in this paper satisfies $\int \phi(x)dx = 1$. Then, we can introduce the following integral form of the Schoenberg operator

$$\tilde{S}f = \sum_{i \in \mathbb{Z}^2} < f\left(\cdot + i\right), \phi > \phi\left(\cdot - i\right),$$

where $< f, \phi >= \int_{\mathbb{R}^2} f\phi$.
Like S, the operator $\tilde{S}$ is also an automorphism on $\mathbb{P}\left(\phi\right)$ and coincides with a difference operator. Indeed, according to Section 3.2, for any $p \in \mathbb{P}\left(\phi\right)$ there exists a unique $q \in \mathbb{P}\left(\phi\right)$ such that $\tilde{S}q = p$. Then,

$$\tilde{S}p = \sum_{i \in \mathbb{Z}^2} < Sq\left(\cdot + i\right), \phi > \phi\left(\cdot - i\right) = \sum_{i \in \mathbb{Z}^2} \left(\sum_{\alpha \in \mathbb{Z}^2} v_\alpha q\left(\alpha + i\right) \right) \phi\left(\cdot - i\right),$$

where $v_\alpha = \int_{\mathbb{R}^2} \phi\left(x\right) \phi\left(x - \alpha\right) dx$. It is easy to see that $v_\alpha = 0$ for all $\alpha \notin \Lambda_k^*$. Then, if we put $\tilde{D}q\left(x\right) = \sum_{\alpha \in H_k^*} v_\alpha q\left(x + \alpha\right)$, we easily verify that we have

$$\tilde{S}p = \sum_{i \in \mathbb{Z}^2} \tilde{D}q\left(i\right) \phi\left(\cdot - i\right) = \sum_{i \in \mathbb{Z}^2} q\left(i\right) \tilde{D}\phi\left(\cdot - i\right) = \tilde{D}Sq = \tilde{D}p.$$

Consequently, $\tilde{S}$ coincides on $\mathbb{P}\left(\phi\right)$ with $\tilde{D}$, and $\tilde{D}^{-1}$ has a finite expression on $\mathbb{P}\left(\phi\right)$.

We now consider the following integral quasi-interpolant based on $\tilde{D}^{-1}$

$$Tf = \tilde{S}\tilde{D}^{-1}f = \sum_{i\in\mathbb{Z}^2} < \tilde{D}^{-1}f\left(\cdot + i\right), \phi > \phi\left(\cdot - i\right)$$
$$= \sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in\Lambda_k^*} b_\alpha < f\left(\cdot + i + \alpha\right), \phi >\right)\phi\left(\cdot - i\right).$$

We notice that for all $p \in \mathbb{P}(\phi)$, we have $Tp = \tilde{S}\tilde{D}^{-1}p = \tilde{D}\tilde{D}^{-1}p = p$. Thus, the iQI T is exact on $\mathbb{P}(\phi)$.
Once again, as we obtained above for the dQI Q,

$$\|T\|_\infty \leq \nu(b) = \sum_{\alpha\in\Lambda_k^*} |b_\alpha|.$$

The study of these iQIs, illustrated by examples, is given in [9], [8] and [14].

Let us denote by $\mathcal{Q}$ one of the above dQI Q or iQI T. It is well known that the infinity norm of $\mathcal{Q}$ appears in the approximation error of f by $\mathcal{Q}f$. More specifically, we have

$$\|f - \mathcal{Q}f\|_\infty \leq \left(1 + \|\mathcal{Q}\|_\infty\right)\operatorname{dist}\left(f, S\left(\phi\right)\right).$$

Then, it is interesting to construct a quasi-interpolant $\mathcal{Q}$ with a small norm. In general, it is difficult to minimize the true norm. To remedy partially this problem, a method for defining discrete quasi-interpolant with minimal infinity norm has been proposed in [14]. It consists in constructing bases of the algebras of lozenge sequences in order to get smaller infinity norms for the corresponding discrete quasi-interpolants. In the next section, we present another method which seems more interesting.

4 Near-best dQIs and iQIs based on Ω-splines

The proposed method consists in choosing a priori a sequence a (resp. b) with a larger support than the biggest lozenge contained in the support of ϕ and afterwards in minimizing $\nu\left(a\right)$ (resp. $\nu\left(b\right)$) under the linear constraints consisting in reproducing all monomials in $\mathbb{P}(\phi)$. More specifically, for $s \geq k$, we construct families of discrete or integral quasi-interpolants

$$Q_{k,l,s}f = \sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in\Lambda_s^*} a_\alpha f\left(i + \alpha\right)\right)\phi\left(\cdot - i\right) \tag{2}$$

$$T_{k,l,s}f = \sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in\Lambda_s^*} b_\alpha < f\left(\cdot + i + \alpha\right), \phi >\right)\phi\left(\cdot - i\right) \tag{3}$$

which satisfy the two following properties:

i) $Q_{k,l,s}$ and $T_{k,l,s}$ are exact on $\mathbb{P}(\phi)$.
ii) The coefficients a_α (resp. b_α), $\alpha \in \Lambda_s^*$, are those that minimize the l_1-norm $\nu(a)$ (resp. $\nu(b)$) of a (resp. b) under the linear constraints consisting in reproducing all monomials in $\mathbb{P}(\phi)$.

As a sequence a (resp. b) is fully determined by a list $\tilde{a} = [a_{\alpha_1}, \cdots, a_{\alpha_n}]$ (resp. $\tilde{b} = [b_{\alpha_1}, \cdots, b_{\alpha_n}]$), it is clear that the exactness of $Q_{k,l,s}$ (resp. $T_{k,l,s}$) on $\mathbb{P}(\phi)$ implies that there exist a $p \times n$ matrix A of rank $p < n$ and a vector d_1 (resp. d_2) in $\mathbb{R}^p$ such that $A\tilde{a} = d_1$ (resp. $A\tilde{b} = d_2$). For $i = 1, 2$, set $V_i = \{\tilde{x} \in \mathbb{R}^n : A\tilde{x} = d_i, i = 1, 2\}$. Then the construction of $Q_{k,l,s}$ or $T_{k,l,s}$ is equivalent to solving the following minimization problem

$$\textbf{Problem (i)} \qquad \text{Solve} \qquad \min\left\{\|x\|_1, \tilde{x} \in V_i\right\}.$$

Definition 8 *If a (resp. b) is a solution of Problem (1) (resp. Problem (2)), then the associated dQI (resp. iQI) defined by 2 (resp. 3) is called a near-best dQI (resp. near-best iQI).*

Proposition 9 *For $i = 1$ or 2, the minimization Problem (i) has at least one solution.*

PROOF. Since the rank of A is p, the above system $A\tilde{x} = d_i, i = 1$ or 2, can be solved and each $x_{\alpha_j}, 1 \le j \le n$, is an affine function of $n - p$ parameters of $\tilde{x}$. Moreover, the sequence x is an element of $\mathcal{L}_k$. On the other hand, by substituting the affine functions x_{α_j} in the expression of $\|x\|_1$, we obtain a $n \times (n-p)$ matrix $\tilde{A}$ and a vector $\tilde{d}_i$ such that $\|x\|_1 = \|\tilde{d}_i - \tilde{A}\tilde{x}\|_1$. Thus, solving Problem (i) is equivalent to determine the best linear l_1-approximation of $\tilde{d}_i$ using the elements of $\tilde{A}\tilde{x}$, and the existence of at least one solution is guaranteed. $\square$

Before giving examples of such quasi-interpolants, note that the exactness equations of $T_{k,l,s}$ on $\mathbb{P}(\phi)$ need the moments $\mu_\alpha(\phi) = \int m_\alpha(x)\,\phi(x)\,dx$, $\alpha \in \Gamma_\phi$, of ϕ. It was shown in [9] that $\mu_\alpha(\phi) = (-\mathbf{i})^{|\alpha|}\mathbf{D}^\alpha\hat{\phi}(\mathbf{0})\mathbf{,}$ $|\alpha| = \alpha_\mathbf{1} + \alpha_\mathbf{2}$. Then, when ϕ is a box-spline, we know explicitly its Fourier transform $\hat{\phi}$ and therefore the computation of $\mu_\alpha(\phi)$ can be easily done. But, for ϕ which is not a box-spline, we can determine its corresponding moments by using only the values $\phi(j)$, $j \in \Lambda_k \cap \mathbb{Z}^2$. Indeed, if we put $t_\alpha = \sum_{j\in\mathbb{Z}^2} m_\alpha(j)\,\phi(j)$, then we have the following result.

Lemma 10 *For any $\alpha \in \Gamma_\phi$ we have*

$$\mu_\alpha(\phi) = \begin{cases} t_\alpha \text{ if } |\alpha| \text{ is even,} \\ 0 \quad \text{otherwise.} \end{cases}$$

PROOF. According to (1), we get the following connection between t_α and g_α:

$$g_\alpha = m_\alpha - \sum_{\beta \leq \alpha, \beta \neq \alpha} \frac{(-1)^{|\alpha-\beta|}\alpha!}{\alpha - \beta} t_{\alpha-\beta} g_\beta. \tag{4}$$

On the other hand (see e.g. [6]) the sequence $(g_\alpha)_{\alpha \in \mathbb{N}^2}$ may be written in the form

$$g_0 = m_0, \qquad g_\alpha = m_\alpha - \sum_{j \in \mathbb{Z}^2} \sum_{\beta \leq \alpha, \beta \neq \alpha} \frac{\alpha!}{\alpha - \beta} (-\mathbf{i}D)^{\alpha-\beta} \hat{\phi}(0)\, g_\beta. \tag{5}$$

Hence, by comparing (4) and (5), we obtain

$$t_\alpha = (-\mathbf{i}D)^{\alpha-\beta} \hat{\phi}(0) = \mu_\alpha(\phi).$$

Using the symmetries of ϕ, we easily verify that $t_\alpha = (-1)^{|\alpha|} t_\alpha$, i.e., $t_\alpha = 0$ for all α such that $|\alpha|$ is odd. Then, the announced result holds. $\square$

5 Examples of near-best dQIs and iQIs

5.1 Near-best dQIs based on quartic box-splines.

Let ϕ be one of the two classical C^2 quartic box-splines $\phi_1 = \sigma * \omega^{1,1}$ and $\phi_2 = \lambda * \omega^{1,1}$. The differential quasi-interpolants based on these B-splines are given by

$$\mathcal{D}f = \sum_{i \in \mathbb{Z}^2} \Big((f(i) - \frac{1}{6} \left(D^{(2,0)} f(i) + D^{(0,2)} f(i) \right)$$
$$+ \frac{1}{720} \left(11 D^{(4,0)} f(i) + 23 D^{(2,2)} f(i) + 11 D^{(0,4)} f(i) \right) \Big) \phi_1(\cdot - i)$$
$$\mathcal{D}f = \sum_{i \in \mathbb{Z}^2} \Big((f(i) - \frac{5}{24} (D^{(2,0)} f(i) + D^{(0,2)} f(i))$$
$$+ \frac{1}{5760} (135 D^{(4,0)} f(i) + 298 D^{(2,2)} f(i) + 135 D^{(0,4)} f(i)) \Big) \phi_2(\cdot - i).$$

In both cases $\mathcal{D}$ is exact on $\mathbb{P}_3$. Then we have the following expressions

$$
\begin{aligned}
m_{0,0} &= \textstyle\sum_{i\in\mathbb{Z}^2}\phi(\cdot - i), & m_{1,0} &= \textstyle\sum_{i\in\mathbb{Z}^2} i_1\phi(\cdot - i),\\
m_{2,0} &= \textstyle\sum_{i\in\mathbb{Z}^2}\left(i_1^2-\mu\right)\phi(\cdot - i), & m_{1,1} &= \textstyle\sum_{i\in\mathbb{Z}^2} i_1 i_2\phi(\cdot - i),\\
m_{3,0} &= \textstyle\sum_{i\in\mathbb{Z}^2}\left(i_1^3-\nu i_1\right)\phi(\cdot - i), & m_{2,1} &= \textstyle\sum_{i\in\mathbb{Z}^2}\left(i_1^2 i_2-\mu i_2\right)\phi(\cdot - i),
\end{aligned}
$$

where

$$
\mu = \begin{cases} \frac{1}{3}, \text{ for } & \phi=\phi_1, \\ \frac{5}{12}, \text{ for } & \phi=\phi_2, \end{cases} \quad \text{and} \quad \nu = \begin{cases} 1, \text{ for } & \phi=\phi_1, \\ \frac{5}{4}, \text{ for } & \phi=\phi_2. \end{cases}
$$

From the above expressions, we deduce by symmetry those of $m_{0,1}$, $m_{0,2}$, $m_{1,2}$ and $m_{0,3}$.
Now, by using the properties of the lozenge sequences $a=(a_\alpha)_{\alpha\in\Lambda_s^*}$, it is easy to verify that the quasi-interpolant

$$
Q_{1,1,s}f = \sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in\Lambda_s^*} a_\alpha f(i+\alpha)\right)\phi(\cdot - i), \ s\geq 1, \tag{6}
$$

is exact on $\mathbb{P}_3$ if and only if the sequence a satisfies the following equations

$$
\textstyle\sum_{\alpha\in\Lambda_s^*} a_\alpha = 1 \quad \text{and} \quad \sum_{\alpha\in\Lambda_s^*} {\alpha_1}^2 a_\alpha = -\mu.
$$

Remark 11 *For $s=1$, the dimension of $\mathcal{L}_1$ coincides with the number of the exactness conditions of $Q_{1,1,1}$ on $\mathbb{P}_3$. Therefore, $Q_{1,1,1}$ is unique and it is given by*

$$
Q_{1,1,1}f = \begin{cases} \sum_{i\in\mathbb{Z}^2}\left(\frac{5}{3}f(i) - \frac{1}{6}\sum_{l=1}^{2} f(i\pm e_l)\right)\phi(\cdot - i), & \text{if } \phi=\phi_1, \\ \sum_{i\in\mathbb{Z}^2}\left(\frac{11}{6}f(i) - \frac{5}{24}\sum_{l=1}^{2} f(i\pm e_l)\right)\phi(\cdot - i), & \text{if } \phi=\phi_2. \end{cases}
$$

In this case, an upper bound of $\|Q_{1,1,1}\|_\infty$ is 2.3333 when $\phi=\phi_1$ and 2.6666 when $\phi=\phi_2$. Thus, in order to have parameters in the minimization problem, it is necessary to take $s>1$. Fig. 2 shows the fundamental functions of the dQIs $Q_{1,1,1}$.

For $s>1$, we distinguish two cases: $s=2t$ and $s=2t+1, t\geq 1$. Assume that $s=2t$ and consider the decomposition of Λ_s^* given by

$$
\Lambda_s^* = I_0^*\cup I_1^*\cup I_2^*\cup\left(\bigcup_{j=2}^{t} I_{2j}^*\right)\cup\left(\bigcup_{j=2}^{t-1} I_{2j+1}^*\right),
$$

where

$$
I_r^* = \left\{l\in\mathbb{Z}^2 : |l_1|+|l_2| = r\right\}.
$$

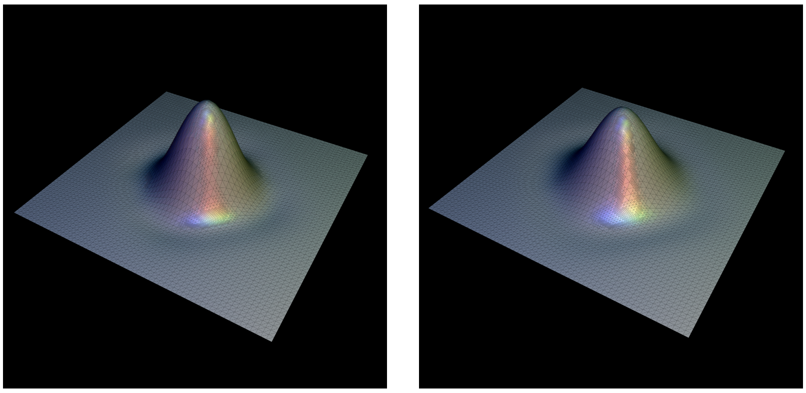

Fig. 2 Fundamental functions of $Q_{1,1,1}$ based on ϕ_1 and ϕ_2.

Let a be the lozenge sequence corresponding to the dQI defined by 6. In order to minimize $\nu(a)$, we get together the components of a following that r is odd or even in I_r^*. Indeed, this regrouping is given by

$$\begin{array}{cccccccccc}
a_{2j,0} & a_{2j-1,1} & a_{2j-2,2} & \cdots & a_{j+1,j} & a_{j,j} & a_{j-1,j+1} & \cdots & a_{2,2j-2} & a_{1,2j-1} \\
a_{0,2j} & a_{-1,2j-1} & a_{-2,2j-2} & \cdots & a_{-j+1,j+1} & a_{-j,j} & a_{-j-1,j-1} & \cdots & a_{-2j+2,2} & a_{-2j+1,1} \\
a_{-2j,0} & a_{-2j+1,-1} & a_{-2j+2,-2} & \cdots & a_{-j-1,-j+1} & a_{-j,-j} & a_{-j+1,-j-1} & \cdots & a_{-2,-2j+2} & a_{-1,-2j+1} \\
a_{0,-2j} & a_{1,-2j+1} & a_{2,-2j+2} & \cdots & a_{j-1,-j-1} & a_{j,-j} & a_{j+1,-j+1} & \cdots & a_{2j-2,-2} & a_{2j-1,-1} \\
(r_1) & (r_2) & (r_3) & & (r_4) & (r_5) & (r_6) & & (r_7) & (r_8)
\end{array}$$

for $I_{2j}^*, j \geq 2$, and by

$$\begin{array}{ccccccccc}
a_{2j+1,0} & a_{2j,1} & a_{2j-1,2} & \cdots & a_{j+1,j} & a_{j,j+1} & \cdots & a_{2,2j-1} & a_{1,2j} \\
a_{0,2j+1} & a_{-1,2j} & a_{-2,2j-1} & \cdots & a_{-j,j+1} & a_{-j-1,j} & \cdots & a_{-2j+1,2} & a_{-2j,1} \\
a_{-2j-1,0} & a_{-2j,-1} & a_{-2j+1,-2} & \cdots & a_{-j-1,-j} & a_{-j,-j-1} & \cdots & a_{-2,-2j+1} & a_{-1,-2j} \\
a_{0,-2j-1} & a_{1,-2j} & a_{2,-2j+1} & \cdots & a_{j,-j-1} & a_{j+1,-j} & \cdots & a_{2j-1,-2} & a_{2j,-1} \\
(\tilde{r_1}) & (\tilde{r_2}) & (\tilde{r_3}) & & (\tilde{r_4}) & (\tilde{r_5}) & & (\tilde{r_6}) & (\tilde{r_7})
\end{array}$$

for $I_{2j+1}^*, j \geq 2$.
Using the properties of symmetry of Λ_s^*, the coefficients in (r_1) (resp. $(\tilde{r_1})$), (r_2) and (r_8) (resp. $(\tilde{r_2})$ and $(\tilde{r_7})$), (r_3) and (r_7) (resp. $(\tilde{r_3})$ and $(\tilde{r_6})$), (r_4) and (r_6) (resp. $(\tilde{r_4})$ and $(\tilde{r_5})$), and in (r_5) are equal.
When $s = 2t+1, t \geq 1$, Λ_s^* can be decomposed in the form

$$\Lambda_s^* = I_0^* \cup I_1^* \cup I_2^* \cup \left(\bigcup_{j=2}^{t} I_{2j}^* \right) \cup \left(\bigcup_{j=2}^{t} I_{2j+1}^* \right),$$

and a similar technique can be used for regrouping the components of the sequence a.

Proposition 12 *Let* $a^*_{0,0} = 1 + \frac{2\mu}{(2t)^2}$ *and* $a^*_{2t,0} = -\frac{\mu}{2(2t)^2}$. *Then*

$$(a^*_{0,0}, \underbrace{0, \cdots, 0}_{t^2+t-1}, a^*_{2t,0}, \underbrace{0, \cdots, 0}_{t})^T \in \mathbb{R}^{(t+1)^2}$$

is a solution of Problem 1 *for* $k = l = 1$ *and* $s = 2t$, $t \geq 1$.

PROOF. For $k = l = 1$ and $s = 2t, t \geq 1$, the expression of $\|a\|_1$ is

$$\begin{aligned}
\|a\|_1 &= |a_{0,0}| + 4|a_{1,0}| + 4(|a_{2,0}| + |a_{1,1}|) + 4|a_{3,0}| + 8|a_{2,1}| \\
&+ 4\sum_{j=2}^{t}(|a_{2j,0}| + |a_{j,j}|) + 8\sum_{j=2}^{t}\sum_{l=1}^{j-1}|a_{2j-l,l}| + 4\sum_{j=2}^{t-1}|a_{2j+1,0}| \\
&+ 8\sum_{j=2}^{t-1}\sum_{l=1}^{j}|a_{2j+1-l,l}| \\
&= |a_{0,0}| + 4|a_{1,0}| + 4\sum_{j=1}^{t}(|a_{2j,0}| + |a_{j,j}|) + 4\sum_{j=1}^{t-1}|a_{2j+1,0}| \\
&+ 8\sum_{j=2}^{t}\sum_{l=1}^{j-1}|a_{2j-l,l}| + 8\sum_{j=1}^{t-1}\sum_{l=1}^{j}|a_{2j+1-l,l}|
\end{aligned}$$

and the associated linear constraints in Problem 1 are

$$\begin{aligned}
1 &= a_{0,0} + 4a_{1,0} + 4\sum_{j=1}^{t}(a_{2j,0} + a_{j,j}) + 4\sum_{j=1}^{t-1}a_{2j+1,0} + 8\sum_{j=2}^{t}\sum_{l=1}^{j-1}a_{2j-l,l} + 8\sum_{j=1}^{t-1}\sum_{l=1}^{j}a_{2j+1-l,l} \\
\alpha &= 2a_{1,0} + 4\sum_{j=1}^{t}j^2(2a_{2j,0} + a_{j,j}) + 2\sum_{j=1}^{t-1}(2j+1)^2 a_{2j+1,0} \\
&+ 4\sum_{j=2}^{t}\sum_{l=1}^{j-1}\left(l^2 + (2j-l)\right)^2 a_{2j-l,l} + 4\sum_{j=1}^{t-1}\sum_{l=1}^{j}\left(l^2 + (2j+1-l)\right)^2 a_{2j+1-l,l}.
\end{aligned} \tag{7}$$

If we put

$$\|a\|_1 = \omega(a_{0,0}, \cdots, a_{2t-1,0}, a_{2t-1,1}, \cdots, a_{2t-1,t-1}, a_{2t,0}, a_{2t,1}, \cdots, a_{2t,t-1}, a_{2t,t}),$$

then, by using equations (7), we can express $a_{0,0}$ and $a_{2t,0}$ in terms of the other coefficients of the lozenge sequence a. Therefore, minimizing $\|a\|_1$ under the linear constraints given in 7 becomes equivalent to minimizing in $\mathbb{R}^{(t+1)^2-2}$ the polyhedral convex function ω of the following variables

$$a_{1,0}, a_{2,0}, a_{2,1}, a_{3,0}, a_{3,1}, \ldots, a_{2t-1,0}, a_{2t-1,1}, \ldots, a_{2t-1,t-1}, a_{2t,1}, \ldots, a_{2t,t-1}, a_{2t,t}. \tag{8}$$

Let $a_{i,j}$ be any variable in (8). Denote by $\bar{\omega}(a_{p,q})$ the restriction of ω obtained by replacing its variables by zero except $a_{p,q}$. We will prove that this univariate function $\bar{\omega}(a_{p,q})$ admits a minimum at $0 \in \mathbb{R}$. Indeed, assume for example

$a_{p,q} = a_{2j,0}$, $1 \leq j \leq t-1$. Then, by annulling the other variables in (7), these equations become

$$a_{0,0} + 4a_{2j,0} + 4a_{2t,0} = 1, \quad \alpha = 4j^2\,(2a_{2j,0}) + 4\left(2t^2\right)a_{2t,0}.$$

Then, the expressions of $a_{0,0}$ and $a_{2t,0}$ in terms of $a_{2j,0}$ are given by

$$a_{0,0} = a^*_{0,0} - 4\left(1 - \frac{j^2}{t^2}\right)a_{2j,0}, \quad a_{2t,0} = a^*_{2t,0} - \frac{j^2}{t^2}a_{2j,0}.$$

Thus, $\bar{\omega}(a_{2j,0})$ takes the following expression

$$\begin{aligned}\bar{\omega}(a_{2j,0}) &= |a_{0,0}| + 4\,|a_{2j,0}| + 4\,|a_{2t,0}| \\ &= \left|a^*_{0,0} - 4\left(1 - \frac{j^2}{t^2}\right)\right| + 4|c_{2j,0}| + 4\left|a^*_{2t,0} - \frac{j^2}{t^2}a_{2j,0}\right|.\end{aligned}$$

In a neighborhood of $a_{2j,0} = 0$, the function $\bar{\omega}\,(c_{2j,0})$ becomes

$$\begin{aligned}\bar{\omega}\,(a_{2j,0}) &= a^*_{0,0} - 4\left(1 - \frac{j^2}{t^2}\right)a_{2j,0} + 4\,|a_{2j,0}| + 4\,(-1)\left(a^*_{2t,0} - \frac{j^2}{t^2}a_{2j,0}\right) \\ &= \omega^* - 4\left(1 - \frac{j^2}{t^2}\right)a_{2j,0} + 4\,|a_{2j,0}| + 4\frac{j^2}{t^2}a_{2j,0},\end{aligned}$$

where $\omega^* = 1 + \frac{\mu}{t^2}$.
When $a_{1,0} > 0$ (resp. $a_{1,0} < 0$), we easily verify that $\bar{\omega}(a_{2j,0}) = \omega^* + \frac{8j^2}{t^2}a_{2j,0} > \omega^*$ (resp. $\bar{\omega}(a_{2j,0}) = \omega^* - 8(1 - \frac{j^2}{t^2})a_{2j,0} > \omega^*$).
A similar technique can be applied for each of the other variables in (8).
Consequently, we conclude that the convex function ω without constraints attains its global minimum at $0 \in \mathbb{R}^{(t+1)^2-2}$. In other words, we have $\omega^* = \omega(a^*_{0,0}, \underbrace{0, \cdots, 0}_{t^2+t-1}, a^*_{2t,0}, \underbrace{0, \cdots, 0}_{t}) = \min\left\{\|a\|_1\,,\ \tilde{a} \in V_1\right\}$. $\square$

Remark 13 *An analogous result can be obtained when s is odd, i.e., $s = 2t+1$, $t \geq 1$. More precisely, if we put $a^*_{0,0} = 1 + \frac{2\mu}{(2t+1)^2}$ and $a^*_{2t+1,0} = -\frac{\mu}{2(2t+1)^2}$, then the vector*

$$\widetilde{a}^* = \left(a^*_{0,0}, 0, \ldots, a^*_{2t+1,0}, 0, \ldots, 0\right)^T \in \mathbb{R}^{(t+1)(t+2)}$$

is asolution of Problem 1 for $k = 1$ and $s = 2t+1$.

According to Proposition 12 and Remark 13, the near minimally normed dQIs associated with Λ^*_s, $s \geq 2$, and exact on $\mathbb{P}_3$ are given by

$$Q_{1,1,s}f = \sum_{i \in \mathbb{Z}^2}\left(\left(1 + \frac{2\mu}{s^2}\right)f\,(i) - \frac{\mu}{2s^2}\sum_{l=1}^{2}f(i \pm se_l)\right)\phi\,(\cdot - i)\,. \tag{9}$$

Proposition 14 *For all $s \geq 1$ we have*

$$\|Q_{1,1,s}\|_{\infty} \leq 1 + 4\frac{\mu}{s^2}.$$

Moreover, the sequence $(Q_{1,1,s})_{s\geq 1}$ converges in the infinity norm to the Schoenberg's operator S.

PROOF. Let $f \in C(\mathbb{R}^2)$ such that $\|f\|_{\infty} \leq 1$. Then, from (9) we get

$$\begin{aligned}
|Q_{1,1,s}f| &\leq \sum_{i\in\mathbb{Z}^2} \left(\left(1 + \frac{2\mu}{s^2}\right) |f(i)| + \frac{\mu}{2s^2} \sum_{l=1}^{2} |f(i \pm s e_l)| \right) \phi(\cdot - i) \\
&\leq \|f\|_{\infty} \sum_{i\in\mathbb{Z}^2} \left(\left(1 + \frac{2\mu}{s^2}\right) + 4\frac{\mu}{2s^2} \right) \phi(\cdot - i) \\
&\leq 1 + 4\frac{\mu}{s^2}.
\end{aligned}$$

Hence, $\|Q_{1,1,s}\|_{\infty} \leq 1 + 4\frac{\mu}{s^2}$.
On the other hand, by using the expression of S given in Section 3.1 , we obtain

$$Q_{1,1,s}f - Sf = \frac{2\mu}{s^2} \sum_{i\in\mathbb{Z}^2} \left(f(i) - \frac{1}{4} \sum_{l=1}^{2} f(i \pm s e_l) \right) \phi(\cdot - i).$$

Therefore

$$|Q_{1,1,s}f - Sf| \leq \frac{2\mu}{s^2} \sum_{i\in\mathbb{Z}^2} (2\|f\|_{\infty}) (\phi(. - i)) \leq \frac{4\mu}{s^2}.$$

Then, we conclude that $\|Q_{1,1,s} - S\|_{\infty} \leq \frac{4\mu}{s^2}$, i.e., $Q_{1,1,s}$ converges to S when $s \longrightarrow +\infty$.

Fig. 3 (resp. 4) shows the fundamental functions of the dQIs $Q_{1,1,s}$ based on ϕ_1 (resp. ϕ_2) for $s = 2, 3$.

Remark 15 *Using the Bernstein-Bézier forms of $\phi = \phi_1$ or ϕ_2, we can easily compute the exact infinity norms of $Q_{1,1,s}$ for the first values of s. For example, if $s = 2$, we get*

$$\begin{aligned}
\|Q_{1,1,2}\|_{\infty} &= \frac{475}{384} \simeq 1.23698 \quad \textit{for} \ \ \phi = \phi_1 \\
\|Q_{1,1,2}\|_{\infty} &= \frac{2009}{1536} \simeq 1.3079 \quad \textit{for} \ \ \phi = \phi_2
\end{aligned}$$

On the other hand, from Proposition 14, we have

$$\|Q_{1,1,2}\|_{\infty} \leq \frac{4}{3} = 1.3333 \ \textit{for} \ \phi = \phi_1 \ \textit{and} \ \|Q_{1,1,2}\|_{\infty} \leq \frac{17}{12} \simeq 1.4166 \ \textit{for} \ \phi = \phi_2.$$

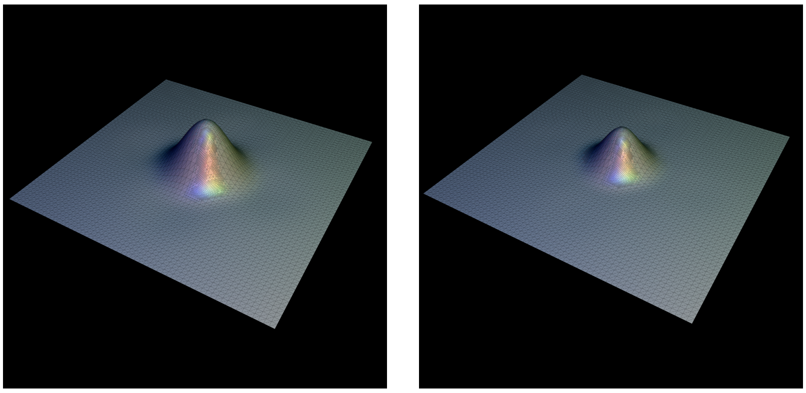

Fig. 3 Fundamental functions of $Q_{1,1,s}$ based on ϕ_1 for $s = 2, 3$.

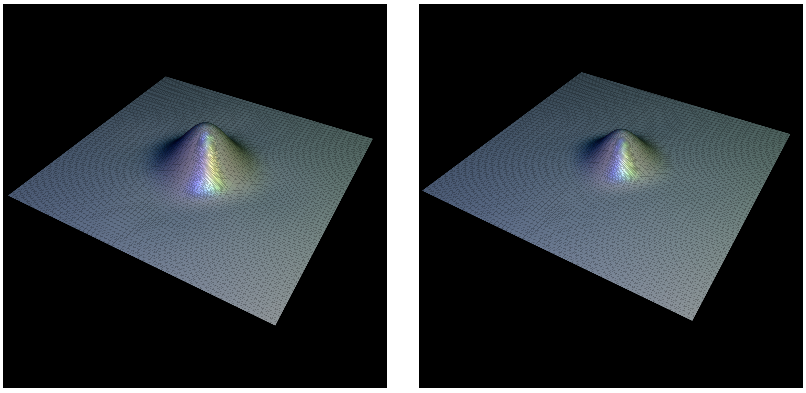

Fig. 4 Fundamental functions of $Q_{1,1,s}$ based on ϕ_2 for $s = 2, 3$.

Therefore, the bounds of $\|Q_{1,1,2}\|_\infty$, are smaller than those of $Q_{1,1,1}$ given in Remark 11. Moreover, these bounds are close to the exact values of the infinity norms of these new dQIs.

5.2 Near-best iQI based on the Ω-spline $\sigma_{\mathbf{1}}^{\mathbf{1,1}}$

According to Section 2, the composed Ω-spline $\sigma_1^{1,1}$ is of class C^4, degree 8 and support $\Omega_{2,1}$. As $\mathcal{S}\left(\sigma_1^{1,1}\right)$ contains polynomials of total degree ≤ 2, one can define quasi-interpolants which are exact on $\mathbb{P}_2$. For instance, by using only the values of $\sigma_1^{1,1}$ on Λ_2^* (see Figure 5), we have got the following expression of its associated differential quasi-interpolant

$$\mathcal{D}f = \sum_{i\in\mathbb{Z}^2} \left(f\left(i\right) - \frac{2}{7}\left(D^{(2,0)} f(i) + D^{(0,2)} f(i) \right) \right) \sigma_1^{1,1}\left(\cdot - i\right).$$

Then we deduce the following formulae

$$m_{0,0}=\sum_{i\in\mathbb{Z}^2}\sigma_1^{1,1}(\cdot-i)\,,\qquad m_{1,0}=\sum_{i\in\mathbb{Z}^2}i_1\sigma_1^{1,1}(\cdot-i)$$
$$m_{2,0}=\sum_{i\in\mathbb{Z}^2}\left(i_1^2-\tfrac{4}{7}\right)\sigma_1^{1,1}(\cdot-i)\,,\ m_{1,1}=\sum_{i\in\mathbb{Z}^2}i_1i_2\sigma_1^{1,1}(\cdot-i)$$

and by symmetry we get the expressions of $m_{0,1}$ and $m_{0,2}$.

$$\begin{array}{ccccc} & & a_{20} & & \\ & a_{11} & a_{10} & a_{11} & \\ a_{20} & a_{10} & a_{00} & a_{10} & a_{20} \\ & a_{11} & a_{10} & a_{11} & \\ & & a_{20} & & \end{array}\qquad \tilde{a}=[\frac{52}{112},\frac{14}{112},0,\frac{1}{112}]$$

Fig. 5. The values of $\sigma_1^{1,1}$ on Λ_2^*.

The near-best iQI based on $\sigma_1^{1,1}$ is given by

$$T_{1,1,s}f=\sum_{i\in\mathbb{Z}^2}\left(\sum_{\alpha\in\Lambda_s^*}b_\alpha<f(\cdot+i+\alpha)\,,\sigma_1^{1,1}>\right)\sigma_1^{1,1}(\cdot-i)\,.$$

From Lemma 10 we deduce the moments $\mu_\alpha=\mu_\alpha\left(\sigma_1^{1,1}\right)=\int_{\mathbb{R}^2}m_\alpha\sigma_1^{1,1}$, $|\alpha|\,|\leq 2$, of $\sigma_1^{1,1}$. Their values are the following

$$\mu_{(0,0)}=1,\quad \mu_{(1,0)}=\mu_{(1,0)}=\mu_{(1,1)}=0,\quad \mu_{(2,0)}=\mu_{(0,2)}=\frac{2}{7}.$$

Then, we easily verify that $T_{1,1,s}$ is exact on $\mathbb{P}_2$ if and only if the coefficients b_α satisfy

$$\sum_{\alpha\in\Lambda_s^*}b_\alpha=1\quad\text{ and }\quad\sum_{\alpha\in\Lambda_s^*}{\alpha_1}^2b_\alpha=-\tfrac{6}{7}.$$

In particular, for $s=1$, these coefficients are unique and the corresponding iQI is given by

$$T_{1,1,1}f=\sum_{i\in\mathbb{Z}^2}\left(\frac{19}{7}<f,\sigma_1^{1,1}>-\frac{3}{7}\sum_{l=1}^{2}<f(\cdot\pm e_l)\,,\sigma_1^{1,1}>\right)\sigma_1^{1,1}(\cdot-i)\,.$$

Now, assume that $s>1$, then by using a similar technique as in Proposition12, one can show the following result.

Proposition 16 *Let* $b_{0,0}^*=1+\frac{12}{7s^2}$ *and* $b_{2t,0}^*=-\frac{3}{7s^2}$. *Then*

$$(b_{0,0}^*,\underbrace{0,\cdots,0}_{t^2+t-1},b_{2t,0}^*,\underbrace{0,\cdots,0}_{t})^T\in\mathbb{R}^{(t+1)^2}$$

is a solution of Problem (2) for $k = 1$ and $s > 1$.

Hence, the near minimally normed iQI based on $\sigma_1^{1,1}$ and exact on $\mathbb{P}_2$ takes the following form

$$T_{1,1,s}f = \sum_{i\in\mathbb{Z}^2}\left(\left(1+\frac{12}{7s^2}\right) < f, \sigma_1^{1,1} > -\frac{3}{7s^2}\sum_{l=1}^{2} < f\left(\cdot \pm e_l\right), \sigma_1^{1,1} >\right)\sigma_1^{1,1}\left(\cdot - i\right).$$

It is simple to check that $\|T_{1,1,s}\|_\infty \leq 1 + \frac{24}{7s^2}$, and therefore the sequence $\left(T_{1,1,s}\right)_{s\geq 1}$ converges in the infinity norm to the operator $\tilde{S}$.

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